18.152 PROBLEM SET 6

due May 10th 11:30 pm.

You can collaborate with other students when working on problems. However, you should write the solutions using your own words and thought.

Problem 1. *For $\Omega = (-\pi, \pi) \subset \mathbb{R}$, show the following statements.*

(1) *$u(x) = |x|$ has a weak derivative.*
(2) *If $u(x) = 1$ for $x > 0$ and $u(x) = 0$ for $x \leq 0$, then $u(x)$ does not have a weak derivative.*
(3) *For $f(x) = \sin(|x|)$, $u(x) = -f(x)$ is not a weak solution to the Dirichlet problem $u'' = f$ in Ω and $u = 0$ on Ω.*
(4) *Let $I_k = (\frac{k-3}{2}\pi, \frac{k-1}{2}\pi)$ where $k = 1, 2, 3$. Suppose $f \in C_c^\infty(\Omega)$, and we have a partition of unity $\varphi_k \in C_c^\infty(I_k)$ with $\sum_{k=1}^{3} \varphi_k = 1$. Assume that $u^k \in C^\infty(\overline{I_k})$ is the smooth solution to $(u^k)'' = f\varphi_k$ in I_k and $u^k = 0$ on ∂I_k. Then, $u = u^1 + u^2 + u^3$ satisfies $u'' = f$ in $\Omega \setminus \{-\frac{\pi}{2}, 0, \frac{\pi}{2}\}$ and $u = 0$ on $\partial\Omega$. However, u is not necessarily a weak solution.*

Hint(2): Suppose $u' = v$, and show $v = 0$ in $\Omega \setminus \{0\}$.
Hint(3): Find a weak derivative u' and use the definition of the weak solution directly with choosing appropriate smooth functions $\varphi \in C_c^\infty$. It'd be helpful to use the integration by parts on each domain where f is smooth.

Problem 2. *Suppose that $u \in C^\infty(\Omega)$ solves $\Delta u = f$ in Ω for $f \in C^\infty(\Omega)$ and $u = 0$ on $\partial\Omega$. Then, given a compact set $K \subset \Omega$, there exists some constant C depending on K, Ω, f, and the (multiple order) derivatives of f such that*

$$\sup_K |u| \leq C.$$

Hint: For some $V \subset \Omega$ containing K, $\|\eta \nabla^k u\|_{L^2(V)}$ for sufficiently large k where $\eta \in C_c^\infty(V)$. Then, apply the Sobolev inequality.

Problem 3. *Let us define $l^2(\mathbb{R}^\infty) = \{v \in \mathbb{R}^\infty : \sum_{i=1}^\infty u_i^2 < +\infty\}$.*

(1) *Show that $\langle v, w \rangle = \sum_{i=1}^\infty v_i w_i$ is an inner product for in $l^2(\mathbb{R}^\infty)$.*
(2) *Show that $l^2(\mathbb{R}^\infty)$ is a Hilbert space.*
(3) *Find a divergent sequence $\{v_i\}_{i=1}^\infty \subset l^2(\mathbb{R}^\infty)$ satisfying $\|v_i\|_{l^2} = 1$.*

Hint(1): To begin with, one must show the well-definedness of $\langle , \rangle$, namely $S_n = \sum_{i=1}^n v_i w_i$ converges as $i \to \infty$. Perhaps, one can obtain this by showing that $\{S_n\}$ is a Cauchy sequence.

Problem 4. *Find a divergent sequence of functions $\{u_i\}$ in $L^2(0, 2\pi)$ such that $\|u_i\|_{L^2(0,2\pi)} = 1$. Then, show that the sequence you find converges weakly in $L^2(0, 2\pi)$.*